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STABILITY ANALYSIS OF COMPLEX STRUCTURES

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Abstract—A systematic approach is presented for the stability analysis of rigid frames and trusses, including the effect of bending moments and shear forces in structures before buckling. The linear graph theory and the transfer matrix technique are employed throughout; the former is perfectly suited for the stability analysis of complex structures, since it automatically takes the geometrical configuration of the "entire" structure into consideration. This permits a derivation of the determining characteristic equation in a general and straightforward fashion. Also, the application of the linear graph theory permits a convenient use of a high-speed digital computer for the numerical computation involved. The present formulation for the stability analysis degenerates into that of the structural analysis in which the effect of axial force on flexure mode is neglected.

INTRODUCTION

IN GENERAL, there are two types of instability (see, for example, [1 and 2]) as shown schematically in Figs. 1(a) and (b), where the load P is plotted against the deformation u at any point of the structure. In Fig. 1(a), the load-deformation diagram consists of three segments OA, AB and AC, with A representing a point of bifurcation. The load P_{cr} is the elastic buckling load. In this type of behavior, the structure has no imperfection and there is no bending moment or shear force in the structure before buckling. In the present paper, this problem is referred to as the problem of Type I. In Fig. 1(b), as soon as the load is applied, the deformation takes place and the instability of the structure occurs when P reaches P_{cr}. This behavior may represent a structure other than that of Type I. This problem is referred to as the problem of Type II. Although a number of papers have been published on stability analysis (see, for example, [3–11]) a systematic approach to the problem of Type II has not been developed.

This paper develops a general, unified approach to the problem of both Type I and Type II, involving linear structures (rigid frames and trusses) with arbitrary plane configuration. The present approach can be extended to structures of three-dimensional configuration without difficulty, although it is not pursued at this time. The determinantal equation for evaluating buckling loads and modes is given explicitly in terms of applied load, structural property and structural configuration. The linear graph theory and the transfer matrix technique are employed throughout the formulation. In particular, the linear

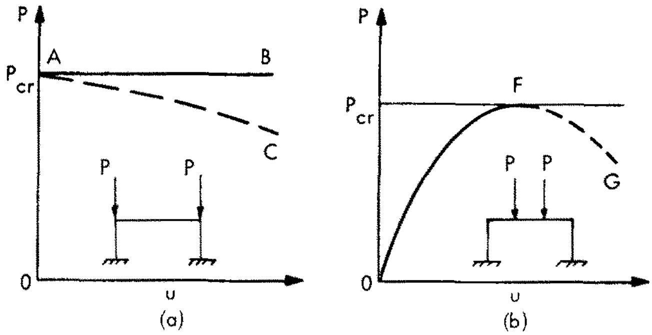

FIG. 1. Load–deformation diagram.

graph theory is perfectly suited for the stability analysis of complex structures, since it automatically takes the geometrical configuration of the "entire" structure into consideration. This permits a derivation of the determining characteristic equation in a general and straightforward fashion. Also, the application of the linear graph theory permits a convenient use of a high-speed digital computer for the numerical computation involved.

It is also shown that the present formulation for the stability analysis degenerates into that of the structural analysis [12–15] in which the effect of axial force on the flexural mode is neglected.

In order to evaluate the buckling load for the problem of Type II, the stress in the structure has to be evaluated as the load increases. This enables one to examine whether the problem should be treated as inelastic buckling, by checking whether the extreme fiber stress in the structure exceeds the yield stress before the elastic buckling load is reached. Several numerical examples are worked out, and the advantage of the present approach is demonstrated.

PRELIMINARY CONSIDERATIONS

Structures considered

Consider an initially stable frame or truss, with two-dimensional arbitrary configuration, consisting of straight members and of supports with no release. Choose the points of support and of intersection of members as nodes. The nodes are identified either by uppercase letters in alphabetical order or by positive integers $1, 2, \ldots, \bar{N}$, with nodes at supports (datum nodes) last, where $\bar{N}$ is the total number of nodes. Number and orient individual members (branches) arbitrarily. Thus a frame or truss is associated with an oriented linear graph [12]. The connectivity of a linear graph can be specified by the branch-node incidence matrix **A** [12–15].

It is assumed that the cross section of each member is uniform for trusses, whereas it can be piecewise uniform for frames, in which case nodes should be created at the points of uniformity change in addition to those at the points of intersection.

Applied loads

When a truss is considered, the externally applied load is limited to a set of concentrated forces acting at the nodes only. For a frame, however, the concentrated forces (including those in the form of a couple) can be applied at any point of the structure. When the concentrated forces are applied to the frame at the points other than the intersections of the members, additional nodes have to be created at the points of application of such loads.

Coordinate systems

Let a global coordinate system fixed in space be denoted by rectangular right-hand axes (ξ, η, ζ), with ξ and η axes defining the plane of the structure. Associated with each branch (member), say branch j, construct a local rectangular right-hand coordinate system (x_j, y_j, z_j) at the center of the initial cross section of that branch. This system is fixed with respect to the global coordinate system in such a way that, in the undeformed state, x_j coincides with the centerline of the branch, and the x_j and y_j axes lie in the plane of the structure.

Branch and nodal quantities

Unless otherwise stated, branch quantities (branch forces and displacements) have a bar if their components are with respect to the global coordinate system, whereas the nodal quantities are always referred to the global coordinate system without a bar.

Let the resultant forces and moment acting on the initial end I and the final end F of branch j of a frame (the end branch forces) be represented by (3×1) vectors with components in the local coordinate system, ${}_I\boldsymbol{\tau}_j = [{}_I\tau_{j1}\ {}_I\tau_{j2}\ {}_I\tau_{j3}]'$, ${}_F\boldsymbol{\tau}_j = [{}_F\tau_{j1}\ {}_F\tau_{j2}\ {}_F\tau_{j3}]'$ in which ${}_I\tau_{j1}$, ${}_I\tau_{j2}$ and ${}_I\tau_{j3}$ are, respectively, the x_j, y_j component of forces (i.e. axial force and shear force) and the bending moment about z_j, at the initial end of the branch. The prime denotes the transpose of a matrix. A similar definition applies to ${}_F\tau_{jk}(k = 1, 2, 3)$ at the final end.

The local components of displacements (including rotation) of the initial and the final ends are the end branch displacements ${}_I\mathbf{u}_j$ and ${}_F\mathbf{u}_j$, ${}_I\mathbf{u}_j = [{}_Iu_{j1}\ {}_Iu_{j2}\ {}_Iu_{j3}]'$, ${}_F\mathbf{u}_j = [{}_Fu_{j1}\ {}_Fu_{j2}\ {}_Fu_{j3}]'$ where ${}_Iu_{j1}$, ${}_Iu_{j2}$ and ${}_Iu_{j3}$ are, respectively, the translations in the x_j, y_j direction and the rotation about z_j of the cross section at the initial end of branch j. A similar definition applies to ${}_Fu_{jk}(k = 1, 2, 3)$ at the final end.

These quantities can be expressed in the global coordinate system by the following transformation: ${}_I\bar{\boldsymbol{\tau}}_j = \mathbf{R}'_j\,{}_I\boldsymbol{\tau}_j$, ${}_I\bar{\mathbf{u}}_j = \mathbf{R}'_j\,{}_I\mathbf{u}_j$ in which $\mathbf{R}_j$ is the (orthogonal) transformation matrix between the global coordinate system and the local coordinate system of branch j.

Let $\boldsymbol{\tau}_j = \boldsymbol{\tau}_j(x_j)$ and $\mathbf{u}_j = \mathbf{u}_j(x_j)$ be (3×1) vectors denoting the branch forces and branch displacements, respectively, at any cross section of the jth branch; i.e. $\boldsymbol{\tau}_j(x_j = 0) = {}_I\boldsymbol{\tau}_j$, $\boldsymbol{\tau}_j(x_j = l_j) = {}_F\boldsymbol{\tau}_j$ where l_j is length of jth branch and $\boldsymbol{\tau}_j = [\tau_{j1}\ \tau_{j2}\ \tau_{j3}]'$, $\mathbf{u}_j = [u_{j1}\ u_{j2}\ u_{j3}]'$. Because of the assumption of no release, the displacements of the end cross sections of those members which meet at a common node are identical. For example, if the node J is the initial node of the branch i and is the final node of the branch j, then ${}_I\bar{\mathbf{u}}_i = {}_F\bar{\mathbf{u}}_j$. Hence, the nodal displacement at the node J, ${}_J\mathbf{u} = [{}_Ju_1\ {}_Ju_2\ {}_Ju_3]'$, is defined as ${}_J\mathbf{u} \equiv {}_I\bar{\mathbf{u}}_i \equiv {}_F\bar{\mathbf{u}}_j$.

Furthermore, introduce a (3×1) vector ${}_J\mathbf{p}$, referred to as the nodal force at the node J, whose elements are the global components of the resultant of forces and couple externally applied at the node J: ${}_J\mathbf{p} = [{}_Jp_1\ {}_Jp_2\ {}_Jp_3]'$ in which ${}_Jp_1$, ${}_Jp_2$ and ${}_Jp_3$ are, respectively, the ξ, η components of the externally applied force and the applied moment about ζ. It is assumed, for simplicity, that ${}_Jp_k(k = 1, 2, 3)$ is either a constant ${}_J\beta_k$ (constant applied force or couple) or ${}_J\beta_k\, P$, where P is an unknown load factor which is to be determined when the structure becomes unstable. The value of P at which the structure becomes unstable is the critical load P_{cr}. The problem involving several independent load factors will be discussed later.

A $(3B \times 1)$ vector $\boldsymbol{\tau}$ and $(3N \times 1)$ vectors $\mathbf{u}$ and $\mathbf{p}$ are then defined as follows: $\boldsymbol{\tau} = [{}_I\boldsymbol{\tau}_1\ {}_I\boldsymbol{\tau}_2 \ldots {}_I\boldsymbol{\tau}_B]'$, $\mathbf{u} = [{}_1\mathbf{u}\ {}_2\mathbf{u} \ldots {}_N\mathbf{u}]'$, $\mathbf{p} = [{}_1\mathbf{p}\ {}_2\mathbf{p} \ldots {}_N\mathbf{p}]'$, where B is the total number of branches and N is the total number of nondatum nodes in the structure.

The quantities defined above can be employed for truss problems with the modification that the last component of ${}_I\boldsymbol{\tau}_j$, ${}_F\boldsymbol{\tau}_j$, ${}_I\mathbf{u}_j$, ${}_F\mathbf{u}_j$, ${}_J\mathbf{u}$ and ${}_J\mathbf{p}$ be disregarded since it is either zero (e.g. ${}_I\tau_{j3}$, ${}_F\tau_{j3}$, etc.) or will be eliminated from the formulation. For example, ${}_I\boldsymbol{\tau}_j = [{}_I\tau_{j1}\ {}_I\tau_{j2}]'$, ${}_I\mathbf{u}_j = [{}_Iu_{j1}\ {}_Iu_{j2}]'$, ${}_J\mathbf{u} = [{}_Ju_1\ {}_Ju_2]'$, ${}_J\mathbf{p} = [{}_Jp_1\ {}_Jp_2]'$, etc.

Sign convention

The standard right-hand rule is adopted as sign convention for the quantities discussed above [12].

FRAMES

Branch equations, transfer matrices and end branch force—displacement equations

Let A_j, I_j, E_j and l_j be the cross-sectional area, the area moment of inertia about the z_j axis, Young's modulus of elasticity and the length, respectively, of the jth branch. The following relationships for the extensional mode of the jth branch can easily be obtained:

$$ {}_F\tau_{j1} = {}_I\tau_{j1} = \tau_{j1} \tag{1} $$

$$ {}_I\tau_{j1} = (A_jE_j/l_j)({}_Fu_{j1} - {}_Iu_{j1}). \tag{2} $$

Equation (1) indicates that the axial force τ_{j1} is constant throughout the branch.

The equation for the flexural mode and those for the branch forces and branch displacement relations can be written as follows:

$$ u''''_{j2} + (\lambda_j^2/l_j^2)u''_{j2} = 0 \tag{3} $$

$$ \lambda_j^2 = -{}_I\tau_{j1}l_j^2/E_jI_j \tag{4} $$

$$ \tau_{j2} = -E_jI_j[u'''_{j2} + (\lambda_j^2/l_j^2)u'_{j2}] \tag{5} $$

$$ \tau_{j3} = E_jI_ju''_{j2} \tag{6} $$

in which the prime denotes the differentiation with respect to x_j.

Using the solution of equation (3) which satisfies the boundary conditions at the initial end, the following transfer equation is obtained [12]:

$$ \begin{bmatrix} {}_F\tau_{j2} \\ {}_F\tau_{j3} \end{bmatrix} = \begin{bmatrix} {}_jf_{11} & {}_jf_{12} \\ {}_jf_{21} & {}_jf_{22} \end{bmatrix} \begin{bmatrix} {}_I\tau_{j2} \\ {}_I\tau_{j3} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & {}_j\bar{f} \end{bmatrix} \begin{bmatrix} {}_Iu_{j2} \\ {}_Iu_{j3} \end{bmatrix} \tag{7}† $$

Furthermore, if the solution of equation (3) which satisfies the displacement boundary conditions at both ends is employed, the following end branch force–displacement relation is obtained [12]:

$$ \begin{bmatrix} {}_I\tau_{j2} \\ {}_I\tau_{j3} \end{bmatrix} = K_{j2} \begin{bmatrix} {}_jg_{11} & {}_jg_{12} \\ {}_jg_{21} & {}_jg_{22} \end{bmatrix} \begin{bmatrix} {}_Iu_{j2} \\ {}_Iu_{j3} \end{bmatrix} + K_{j2} \begin{bmatrix} {}_j\bar{g}_{11} & {}_j\bar{g}_{12} \\ {}_j\bar{g}_{21} & {}_j\bar{g}_{22} \end{bmatrix} \begin{bmatrix} {}_Fu_{j2} \\ {}_Fu_{j3} \end{bmatrix} \tag{8}† $$

† In what follows, the definitions of undefined quantities or symbols in the equations with a dagger are given in Appendix 1.

The transfer equations given in equations (1) and (7) and the end branch force–displacement relations given in equations (2) and (8) can be written in the following matrix forms:

$$ {}_F\boldsymbol{\tau}_j = \mathbf{C}_j\, {}_I\boldsymbol{\tau}_j + \mathbf{D}_j\, {}_I\mathbf{u}_j \tag{9†} $$

$$ {}_I\boldsymbol{\tau}_j = \mathbf{K}_j(\mathbf{G}_j\mathbf{R}_j\, {}_I\bar{\mathbf{u}}_j + \bar{\mathbf{G}}_j\mathbf{R}_j\, {}_F\bar{\mathbf{u}}_j) \tag{10†} $$

in which $\mathbf{K}_j$ is a (3×3) diagonal matrix with elements K_{j1}, K_{j2} and K_{j3} where $K_{j1} = A_jE_j/l_j$, and $K_{j2} = K_{j3} = E_jI_j/l_j^2$.

When the effect of axial force on flexural mode is neglected or the axial force in branch j is zero, one can reduce the transcendental elements in $\mathbf{C}_j$, $\mathbf{D}_j$, $\mathbf{G}_j$ and $\bar{\mathbf{G}}_j$ to constants by taking a limit as λ_j approaches zero. The resulting matrices $\underline{\mathbf{C}}_j$, $\underline{\mathbf{G}}_j$ and $\underline{\bar{\mathbf{G}}}_j$ thus contain no transcendental element, and ${}_j\bar{f} = 0$ so that $\underline{\mathbf{D}}_j = 0$.

Nodal equations

Using equation (9), the equation of equilibrium at a nondatum node, say node J, can be written as

$$ {}_J\mathbf{p} - \sum_k \mathbf{D}_k\, {}_I\bar{\mathbf{u}}_k = -\sum_i \mathbf{R}_i'\, {}_I\boldsymbol{\tau}_i + \sum_k \mathbf{R}_k'\mathbf{C}_k\, {}_I\boldsymbol{\tau}_k \tag{11} $$

where the index i refers to those branches positively incident on node J, while k refers to those branches negatively incident on J.

System equations

Define system matrices $\mathbf{K}$, $\bar{\mathbf{Q}}$, $\mathbf{Q}$ and $\mathbf{Y}$, each element of which is a (3×3) matrix of the individual branch quantities or of the individual nodal quantities, as follows:

$$ \mathbf{K} = [\mathbf{K}_j], j = 1, 2, \ldots B \tag{12} $$

$$ \bar{\mathbf{Q}} = [\bar{\mathbf{q}}_{iJ}], i = 1, 2, \ldots, B, J = 1, 2, \ldots N \tag{13} $$

$$ \mathbf{Q} = [\mathbf{q}_{iJ}], i = 1, 2, \ldots, B, J = 1, 2, \ldots, N \tag{14} $$

$$ \mathbf{Y} = [\mathbf{y}_{IJ}], I = 1, 2, \ldots, N, J = 1, 2, \ldots N \tag{15} $$

$$ \bar{\mathbf{q}}_{iJ} = \begin{bmatrix} 0 & \text{if } a_{iJ} = 0 \\ \mathbf{G}_i\mathbf{R}_i & \text{if } a_{iJ} = 1 \\ \bar{\mathbf{G}}_i\mathbf{R}_i & \text{if } a_{iJ} = -1 \end{bmatrix} \tag{16} $$

$$ \mathbf{q}_{iJ} = \begin{bmatrix} 0 & \text{if } a_{iJ} = 0 \\ -\mathbf{R}_i & \text{if } a_{iJ} = 1 \\ \mathbf{C}_i'\mathbf{R}_i & \text{if } a_{iJ} = -1 \end{bmatrix} \tag{17} $$

$$ \mathbf{y}_{IJ} = \begin{bmatrix} -\mathbf{D}_m & \text{if } e_{IJ} = 1 \text{ and } m \text{ is the branch} \\ & \text{connecting nodes } I \text{ and } J \\ 0 & \text{otherwise} \end{bmatrix} \tag{18} $$

where a_{iJ} is the $i-J$ element of branch-node incidence matrix $\mathbf{A}$ [12] and where e_{IJ} is the $I-J$ element of the node-node incidence matrix $\mathbf{E}$ [12]. Matrices $\mathbf{Q}$ and $\bar{\mathbf{Q}}$ are called modified branch-mode incidence matrices and $\mathbf{Y}$, the modified node-node incidence matrix.

With the aid of these system matrices, the end branch force–displacement relations and the equations of equilibrium at nondatum nodes of the system can be derived from equations (10) and (11) as follows:

$$\boldsymbol{\tau} = \mathbf{K}\bar{\mathbf{Q}}\mathbf{u} \tag{19}$$

$$\mathbf{Q}'\boldsymbol{\tau} = \mathbf{p}+\mathbf{Y}\mathbf{u} \tag{20}$$

Substitution of equation (19) into equation (20) yields

$$(\mathbf{Q}'\mathbf{K}\bar{\mathbf{Q}}-\mathbf{Y})\mathbf{u} = \mathbf{p} \tag{21}$$

It should be mentioned that $\bar{\mathbf{Q}}$, $\mathbf{Q}$ and $\mathbf{Y}$ are functions of λ_j and hence function of axial forces ${}_j\tau_{j1}$, $j = 1, 2, \ldots, B$, and equations (19) and (20) are the formulation for the structural analysis in which the effect of axial force on flexural mode is taken into account.

Buckling loads and buckling modes

It can be shown from equations (19) and (20) that the criterion for instability is the vanishing of the determinant $|\mathbf{Q}'\mathbf{K}\bar{\mathbf{Q}}-\mathbf{Y}|$; i.e.

$$|\mathbf{Q}'\mathbf{K}\bar{\mathbf{Q}}-\mathbf{Y}| = 0 \tag{22}$$

The values of P (eigenvalues) at which equation (22) is satisfied are the buckling loads, while the corresponding displacement eigenvector $\Delta\mathbf{u}$ are the associated buckling modes.

Location of eigenvalues of the determinantal (characteristic) equation $|\mathbf{Q}'\mathbf{K}\bar{\mathbf{Q}}-\mathbf{Y}| = 0$

Both Type I and Type II problems are considered below.

Problem of Type I [*Fig.* 1(a)]. It is noted that for Fig. 1(a) to be possible, the axial deformation of the branch should be disregarded. Hence, the problem of Type I refers to those problems for which, when the axial deformation is neglected, no deformation occurs before the critical load is reached, and for which the axial force in each branch, τ_{j1}, $j = 1, 2, \ldots, B$, is known explicitly in terms of the load factor P. Thus, the values of the determinant $|\mathbf{Q}'\mathbf{K}\bar{\mathbf{Q}}-\mathbf{Y}|$ are computed numerically for a series of values of $\mathbf{P}$ (usually with a small, identical increment ΔP) within a range where the first several buckling loads are located. The fundamental buckling load occurs in a region where the determinantal value first changes its sign from positive to negative. Then the interpolation can be used to find the buckling load using several determinantal values in this region. Sometimes, the calculations of determinantal values have to be performed twice, first by using a relatively large increment of ΔP so as to locate the approximate vicinity of buckling load and then using a smaller increment ΔP in this region for the purpose of interpolation.

Once the eigenvalue has been calculated, its associated displacement eigenvector $\Delta\mathbf{u}$ (buckling mode) can be evaluated numerically.

Problem of Type II. To compute the determinant $|\mathbf{Q}'\mathbf{K}\bar{\mathbf{Q}}-\mathbf{Y}|$, axial forces τ_{j1}, $j = 1, 2, \ldots, B$ should first be evaluated iteratively, using the following expression obtained from equations (19) and (21):

$$\boldsymbol{\tau} = \mathbf{K}\bar{\mathbf{Q}}(\mathbf{Q}'\mathbf{K}\bar{\mathbf{Q}}-\mathbf{Y})^{-1}\mathbf{p} \tag{23}$$

since axial forces are unknown for the problem of Type II. The successive approximation procedure is described below.

The iterative procedure is initiated by neglecting the axial forces τ_{j1}, $j = 1, 2, \ldots, B$, appearing on the right-hand side of equation (23) so that the first estimate $\boldsymbol{\tau}^{(1)}$ of $\boldsymbol{\tau}$ can be calculated. Note that this first estimate is equivalent to the classical structural analysis [12, 15]. Since the $(3j-2)th$ elements of $\boldsymbol{\tau}^{(1)}$, $j = 1, 2, \ldots, B$, represent the axial forces in branches, those values are then employed on the right-hand side of equation (23) to obtain the second estimate $\boldsymbol{\tau}^{(2)}$. After a few cycles of iteration, the branch forces $\boldsymbol{\tau}$ and the determinantal value $|\mathbf{Q}'\mathbf{K}\bar{\mathbf{Q}} - \mathbf{Y}|$ converge, if P is not very close to the buckling load. This iterative procedure enables one to locate a narrow region of P within which the buckling load is located. When P approaches the buckling load, however, the procedure above may exhibit erratic oscillation. The region of P in which the iterative procedure does not converge is referred to as the oscillating region, and the upper bound and the lower bound of this region are the upper bound and the lower bound of the buckling load.

At this point, some alternative procedures can be taken as follows: (a) one can estimate the buckling load by interpolating the determinantal values evaluated outside the oscillating region, (b) the determinantal values within the oscillating region can be estimated, in approximation, by using the axial forces of the first estimates $\tau^{(1)}$, or (c) the oscillating region may further be narrowed by the iterative procedure, starting with the solution due to $P - \Delta P$ instead of starting with the classical structural analysis. Any alternatives or combination of them can be used, depending on the individual problem.

It is clear that during the process of iteration, the branch force vector $\boldsymbol{\tau}$ is calculated from equation (23). This enables one to examine whether the structure considered will fail due to elastic buckling, by checking whether the fiber stress of the structure exceeds the yield stress of the structural material before the critical load is reached.

Releases

The preceding discussion is based upon the assumption that no release exists within the frame. If, however, the releases occur at either supports (datum nodes) (including hinges, rollers or elastic constraints) or at interior nodes, the present formulation is still valid, with a slight modification of the end branch force–displacement relation [equation (8)] in a similar fashion as discussed in Ref. [12].

Structural analysis

The same pair of equations [equations (19) and (20)] are valid for the classical structural analysis of frames in which the effect of axial force on flexure mode is neglected, and hence matrices $\underline{\mathbf{C}}_j$, $\underline{\mathbf{G}}_j$ and $\underline{\bar{\mathbf{G}}}_j$ should be used, with $\underline{\mathbf{D}}_j = 0$, $\mathbf{Y} = 0$ [12, 15].

Buckling without nodal translation

The formulation of the stability criterion for the problem of Type I can be simplified considerably if (a) the axial deformation of the branch is neglected and (b) no nodal translation (sideways) occurs during buckling. Since, in this case, ${}_ju_1 = {}_ju_2 = 0$, we define ${}_j\mathbf{u} = {}_ju_3$, $\boldsymbol{\tau}_j = {}_I\tau_{j3}$, $\mathbf{R}_j = 1$ and take into account only the third equation of equations (10) and (11). Then, the determinantal equation for the critical load $|\mathbf{Q}'\mathbf{K}\bar{\mathbf{Q}} - \mathbf{Y}| = 0$ is still valid except that the dimension of this determinant reduces to $N \times N$. It is clear, in this case, that $\mathbf{K}_j = K_{j2}$, $\mathbf{G}_j = {}_jg_{22}$, $\bar{\mathbf{G}}_j = {}_j\bar{g}_{22}$, $\mathbf{C}_j = {}_jf_{22}$ and $\mathbf{D}_j = {}_j\bar{f}$. For the branch in which the axial force is zero, one has $\underline{\mathbf{G}}_j = -4l_j$, $\underline{\bar{\mathbf{G}}}_j = -2l_j$, $\underline{\mathbf{C}}_j = 1$ and $\underline{\mathbf{D}}_j = 0$.

Critical surface

To simplify the presentation, the discussion above is based on the assumption that there is only one load factor P to be determined for the critical value. If there are m independent load factors, $P_1, P_2, \ldots, P_m$, the approach discussed can be applied to evaluate numerically a critical surface (e.g. Ref. [8]) defining the stable region, in the m-dimensional space in which each load factor P_j corresponds to one coordinate axis.

TRUSSES

As pointed out previously, only the first two components of ${}_I\boldsymbol{\tau}_j$, ${}_F\boldsymbol{\tau}_j$, ${}_I\mathbf{u}_j$, ${}_F\mathbf{u}_j$, ${}_J\mathbf{p}$, ${}_J\mathbf{u}$, etc., are needed for the solution of the truss problem. Although ${}_I\mathbf{u}_j$ and ${}_F\mathbf{u}_j$ actually have three components, the last component is eliminated from the evident condition that hinges cannot resist couples. The branch differential equations and the branch force–displacement relations are the same as those associated with frames.

Branch equations

Equations (1) and (2) for the extensional mode are still valid. However, in deriving the branch equations for the flexural mode, the boundary conditions $u''_{j2}(0) = 0$, $u''_{j2}(l_j) = 0$, should be used. Thus, in a similar fashion to frame problem, one obtains the transfer matrix and end branch force–displacement relationships for jth branch as follows:

$$ {}_F\boldsymbol{\tau}_j = {}_I\boldsymbol{\tau}_j \tag{24} $$

$$ {}_I\boldsymbol{\tau}_j = \mathbf{K}_j\boldsymbol{\Lambda}_j({}_F\bar{\mathbf{u}}_j - {}_I\bar{\mathbf{u}}_j) \tag{25} $$

where $\boldsymbol{\Lambda}_j$ is a (2×2) orthogonal transformation matrix from global coordinate system (ξ, η) to local coordinate systems (x_j, y_j) of branch j, and $\mathbf{K}_j$ is a (2×2) diagonal matrix with elements A_jE_j/l_j and ${}_I\tau_{j1}/l_j$.

It is noted that in applying the boundary conditions $u''_{j2}(0) = 0$ and $u''_{j2}(l_j) = 0$ to the solution of equation (3), one arrives at a possible condition:

$$ \tau_{j1} = n^2\pi^2 E_jI_j/l_j^2\,;\; n = 1, 2, 3, \ldots \tag{26} $$

which indicates the Euler buckling load of branch j. If this condition is satisfied, the displacement $u_{j2}(x_j)$ is not unique, implying the instability of the jth branch and hence the instability of the structure. Therefore, with the assumption that the branch axial force τ_{j1} does not exceed the Euler buckling load, equations (24) and (25) are used in the following formulation for system instability. It should be emphasized that such a distinction does not exist between individual branch instability and system instability in a frame problem, because the end of each branch of a frame is subjected to bending moment constraints from other branches so that buckling of an individual branch can not occur independently.

Nodal equations

Using equation (24), the equation of equilibrium at a nondatum node, say node J, can be written as

$$ \sum_k \boldsymbol{\Lambda}'_k\, {}_I\boldsymbol{\tau}_k - \sum_i \boldsymbol{\Lambda}'_i\, {}_I\boldsymbol{\tau}_i = {}_J\mathbf{p} \tag{27} $$

where the index i refers to those branches positively incident on node J while k refers to those branches negatively incident on node J.

System equations

Define system matrices **K** and **Q**, each element of which is a matrix of the individual branch quantities or of the individual nodal quantities, as follows:

$$\mathbf{K} = [\mathbf{K}_j], j = 1, 2, \ldots, B \tag{28}$$

$$\mathbf{Q} = [\mathbf{q}_{iJ}], i = 1, 2, \ldots, B, J = 1, 2, \ldots N \tag{29}$$

$$\mathbf{q}_{iJ} = \begin{bmatrix} 0 & \text{if } a_{iJ} = 0 \\ -\mathbf{\Lambda}_i & \text{if } a_{iJ} = 1 \\ \mathbf{\Lambda}_i & \text{if } a_{iJ} = -1 \end{bmatrix} \tag{30}$$

where, again, a_{iJ} is the $i-J$ element of the branch-node incidence matrix **A** [12].

With the aid of these system matrices, the end branch force–displacement equations and the equations of equilibrium at nondatum nodes of the system can be derived from equations (25) and (27) as follows:

$$\boldsymbol{\tau} = \mathbf{KQu} \tag{31}$$

$$\mathbf{Q}'\boldsymbol{\tau} = \mathbf{p} \tag{32}$$

Only the matrix **K** is a function of axial force ${}_I\tau_{j1}, j = 1, 2, \ldots B$.

Buckling loads and buckling modes

It can be shown from equations (31) and (32) that the system instability criterion is

$$|\mathbf{Q}'\mathbf{KQ}| = 0 \tag{33}$$

The eignvalues of equation (33) are the buckling loads, while the associated displacement eigenvectors $\Delta\mathbf{u}$ are the corresponding buckling modes. The techniques discussed in the frame problem for evaluating the buckling loads can be employed here for the truss problem.

It is emphasized hère again that the buckling load calculated from equation (33) will be of practical interest only when the axial force in each branch does not exceed Euler's buckling load, since otherwise the system will fail due to buckling of individual branches and not as a result of system instability.

Releases

When the support is a roller, it should be replaced by a fictitious branch, for the analysis to be consistent with the general formulation described previously [12]. The extensional stiffness of this branch AE/l is set to be very large in comparison with that of other branches.

Structure analysis

When the effect of axial force on flexural mode is neglected, it follows that $\tau_{j2} = 0$. Hence, if we define ${}_I\tau_j = \tau_{j1}$, the same pair of equations (31) and (32) are valid for the structural analysis of trusses [12, 14].

NUMERICAL EXAMPLES

To compute numerical values of buckling loads and buckling modes, a general computer program is written in which equations (22) and (33) are used for problems of Type I and equation (23) is employed for problems of Type II in addition to equation (22). As can be realized from the preceding formulation, this computer program is capable of analyzing two-dimensional frames and trusses of any geometrical configuration. It can also perform the classical structural analysis described in the preceding sections.

In the following examples, Young's modulus of elasticity is assumed to be the same for each member and is denoted by E.

Example 1

The fundamental buckling load of a portal frame shown in Fig. 2(a), with $I_1 = I_2 = I_3$, $l_1 = l_2 = 10$ in., $A_1 = A_2 = A_3$, $r_1^2 = r_2^2 = r_3^2 = 1{\cdot}0$ in.^{2} is found to be $6{\cdot}979\ EI_1/l_1^2$, where r_j is the radius of gyration of the jth branch. The corresponding normalized buckling mode $\Delta\mathbf{u}$ is shown in Fig. 2(b). The fundamental buckling load of the same problem is found to be $7{\cdot}4\ EI_1/l_1^2$ (see also Ref. [2], pp. 327–332) in which the axial deformation is neglected.

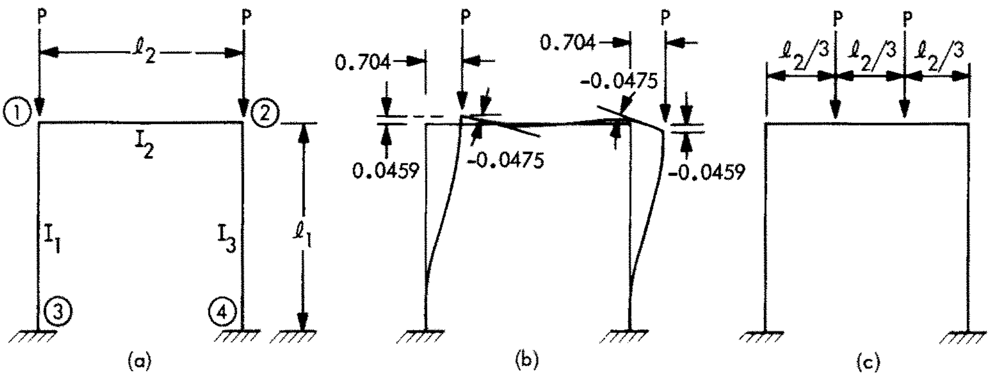

FIG. 2. Portal frame.

Example 2

The fundamental buckling load of the same problem as Example 1 is found to be $7{\cdot}349\ EI_1/l_1^2$ for $l_1 = l_2 = l_3 = 120$ in., $r_1^2 = r_2^2 = r_3^2 = 10$ in.^{2}.

Example 3

The same frame structure as that given in Example 2 but loaded at one third of the beam as shown in Fig. 2(c) is considered. The fundamental buckling load is found to be 7·30 EI_1/l_1^2, which is lower than the fundamental buckling load of Example 2. Since this belongs to the problem of Type II, the iterative procedure discussed previously is employed. The solution is obtained by the interpolation using the determinantal values evaluated at $P = 7{\cdot}28\ EI_1/l_1^2, 7{\cdot}285, 7{\cdot}290, 7{\cdot}295, \ldots 7{\cdot}325\ EI_1/l_1^2$. No oscillation is observed for iteration. For this particular problem, if the material is made of steel, the yielding stress in the structure occurs before the fundamental buckling load is reached, and therefore, $7{\cdot}30\,EI_1/l_1^2$ has no practical interest except to demonstrate the validity of the approach discussed in this paper.

Example 4

The fundamental buckling load of a gable frame shown in Fig. 3(a) is obtained as $5{\cdot}334\,EI/l^2$ with ${}_2\beta_2 = {}_4\beta_2 = 0$. The area of each member is proportional to its moment of inertia and $r^2 = 10$ in.^{2} for each member, where r is the radius of gyration. The buckling mode $\mathbf{\Delta u}$ is also plotted in Fig. 3(b).

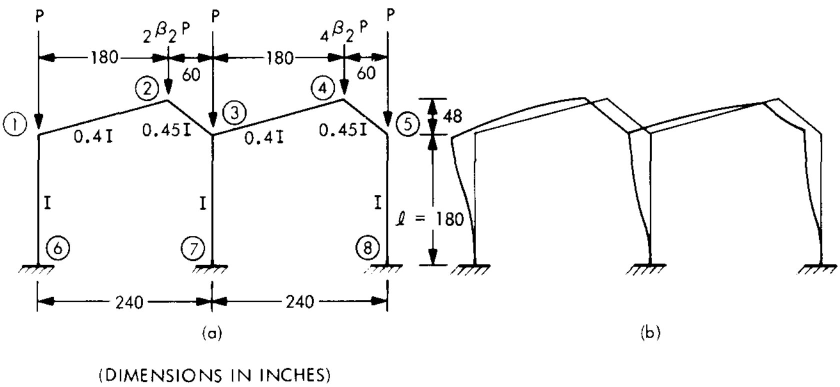

FIG. 3. Gable frame.

Example 5

The fundamental buckling load of Example 4 with ${}_2\beta_2 = 0{\cdot}05$ and ${}_4\beta_2 = 0{\cdot}05$ [Fig. 3(a)] is considered. Since this belongs to the problem of Type II, the iterative procedure discussed previously is employed. For this particular example, the iterative procedure for evaluating the determinantal values oscillates considerably within the range $P = 5{\cdot}04\,EI/l^2$ and $P = 5{\cdot}3\,EI/l^2$ and the determinantal value changes sign within this region. Hence, $5{\cdot}3\,EI/l^2$ and $5{\cdot}04\,EI/l^2$ are, respectively, the upper and the lower bound of the fundamental buckling load. The determinantal values within this oscillating region are estimated, in approximation using the axial force of $\boldsymbol{\tau}^{(1)}$ and then the fundamental buckling load is estimated as $5{\cdot}157\,EI/l^2$. This buckling load is lower than that of Example 4.

Example 6

The first three buckling loads of a two-bay three-story nonsymmetric building structure [Fig. 4(a)] are obtained as $5{\cdot}004\,EI/l^2$, $6{\cdot}851\,EI/l^2$ and $9{\cdot}760\,EI/l^2$, where the area of each member is proportional to its moment of inertia and the radius of gyration for each member is 5·0 in. The first three buckling modes are plotted in Figs. 4(b), (c) and (d), respectively.

Example 7

The fundamental buckling load for a truss shown in Fig. 5(a) is found to be $0{\cdot}17157\,AE$, with $b = l/2 = 50$ in., $A_1 = A$. This buckling load is the same as that given in Ref. [16] (p. 148), where the critical load is expressed as $P_{cr} = AE/\cot^3\alpha_2[3+(2Al_1/A_1 l\cos^2\alpha_2)]$,

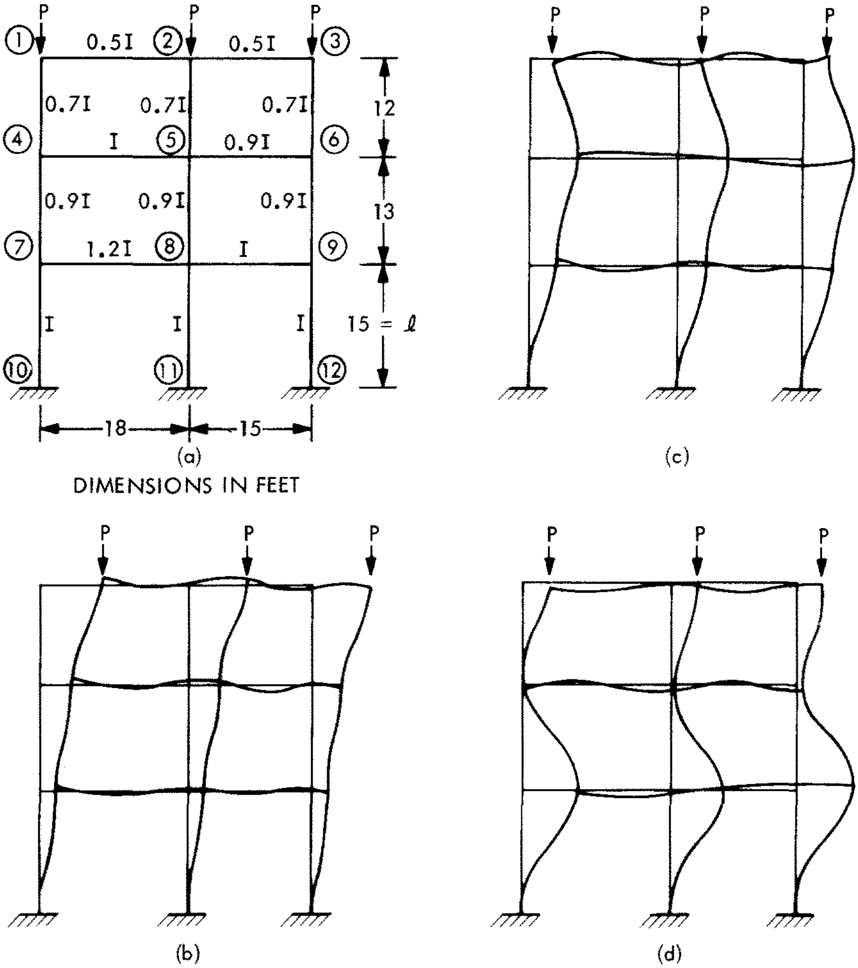

FIG. 4. Multistory structure.

with α_2 being the angle between the diagonal member and the cord member. The buckling mode is also plotted in Fig. 5(b).

Example 8

The fundamental buckling load for a truss shown in Fig. 5(c) is found to be 0·1336 AE, where $A_1 = 1{\cdot}5\,A$, $A_2 = 1{\cdot}2\,A$ and $b = l = 100$ in. The associated buckling mode is plotted in Fig. 5(d). It is mentioned here again that the buckling loads given in Examples 7 and 8 are of practical interest if the axial force in each member does not exceed the Euler buckling load.

CONCLUSION

A systematical unified method is presented for the stability analysis of complex structures, including the effect of bending moments and shear forces in structures before buckling.

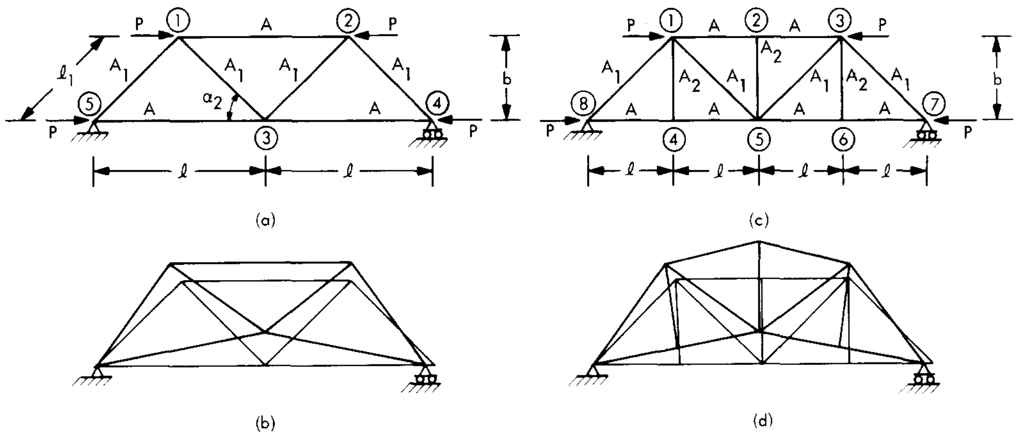

FIG. 5. Truss structures.

The determinantal (characteristic) equation for evaluating the buckling load is given, explicitly, in terms of structural property and structural geometric configuration for rigid frames and trusses. Network concept and transfer matrix technique are employed throughout the formulation so that the geometric configurations of structures are taken into account in a general fashion. This permits a convenient use of a high-speed digital computer for the numerical work involved in the analysis. It is shown that the general formulation degenerates into that of structural analysis when the effect of axial force of flexural mode is neglected. The formulation is particularly suitable for the stability analysis of complex structures.

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APPENDIX 1—DEFINITIONS

In which ${}_jf_{11} = 1$, ${}_jf_{12} = 0$, ${}_jf_{21} = -l_j(\sin\lambda_j)/\lambda_j$, ${}_jf_{22} = \cos\lambda_j$, ${}_j\bar{f} = -E_jI_j\lambda_j(\sin\lambda_j)/l_j$, ${}_jg_{11} = (\lambda_j\sin\lambda_j)/l_jF_j$, ${}_jg_{12} = (1-\cos\lambda_j)/F_j$, ${}_jg_{21} = {}_jg_{12}$, ${}_jg_{22} = (\sin\lambda_j - \lambda_j\cos\lambda_j)l_j/\lambda_jF_j$, ${}_j\bar{g}_{11} = -\lambda_j(\sin\lambda_j)/l_jF_j$, ${}_j\bar{g}_{12} = {}_jg_{12}$, ${}_j\bar{g}_{21} = -{}_j\bar{g}_{12}$, ${}_j\bar{g}_{22} = l_j(\lambda_j - \sin\lambda_j)/\lambda_jF_j$, $F_j = [\lambda_j\sin\lambda_j - 2(1-\cos\lambda_j)]/\lambda_j^2$.

$$\mathbf{C}_j = \begin{bmatrix} 1 & 0 & 0 \\ 0 & {}_jf_{11} & {}_jf_{12} \\ 0 & {}_jf_{21} & {}_jf_{22} \end{bmatrix}, \mathbf{D}_j = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & {}_j\bar{f} \end{bmatrix}$$

$$\mathbf{G}_j = \begin{bmatrix} -1 & 0 & 0 \\ 0 & {}_jg_{11} & {}_jg_{12} \\ 0 & {}_jg_{21} & {}_jg_{22} \end{bmatrix}, \bar{\mathbf{G}}_j = \begin{bmatrix} 1 & 0 & 0 \\ 0 & {}_j\bar{g}_{11} & {}_j\bar{g}_{12} \\ 0 & {}_j\bar{g}_{21} & {}_j\bar{g}_{22} \end{bmatrix}$$

$$\underline{\mathbf{C}}_j = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -l_j & 1 \end{bmatrix}, \underline{\mathbf{G}}_j = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -12/l_j & -6 \\ 0 & -6 & -4l_j \end{bmatrix},$$

$$\underline{\bar{\mathbf{G}}}_j = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 12/l_j & -6 \\ 0 & 6 & -2l_j \end{bmatrix}$$

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Абстракт—Дается систематический подход к решению задачи устойчивости для жестких рам и ферм, учитывающий эффект изгибных моментов и сил сдвига, в конструкциях до момента потери устойчивости. Используется, с начала до конца, теория линейных графов и способ матрицы перемещений. Эта последняя идеально пригодна для анализа устойчивости сопряженных конструкций, так как автоматически рассматривает конфигурацию всей конструкции. По этому можно вывести определяющее характеристическое уравнение общим и простым способом. Применение линейной теории графов позваляет, также удобно использовать ЭЦВМ для сложных расчетов. Предлагаемая формулировка для анализа устойчивости позваляет на такой анализ конструкций, в котором пренебрегается эффектом осевых усилий на режим изгиба.